STOR 831: WEAK CONVERGENCE AND LOCAL WEAK CONVERGENCE
FALL 2015, TU-TH: 12:30-1:45 (HANES 125).
INSTRUCTOR: SHANKAR BHAMIDI

1. INTRODUCTION

One of the central themes in both the theory and application of probability is understanding asymptotics of a sequence of random objects $\{X_n : n \geqslant 1\}$ as $n \to \infty$. In undergraduate probability as well as a first graduate level course in probability, much of the emphasis is on real valued random variables. As we progresses towards the forefront of research, we come across much more complicated objects including:

(a) Random objects in function space representing empirical processes derived from a sequence of observations [16] or lengths of queues [17].
(b) Random probability measures representing the spectral distribution of a large random matrix [5].
(c) Random metric spaces describing large trees (e.g. phylogenetic trees or the minimal spanning tree) or network models in probabilistic combinatorics [1, 2, 13].

Coupled with the above examples, in the last decade a new notion of convergence called *Local weak convergence* has started to play a fundamental role in the probabilistic study of a wide range of problems. The main idea is as follows: When studying large discrete structures (for example a graphical model on a large graph), if we study "local neighborhoods" of a uniformly chosen random vertex,

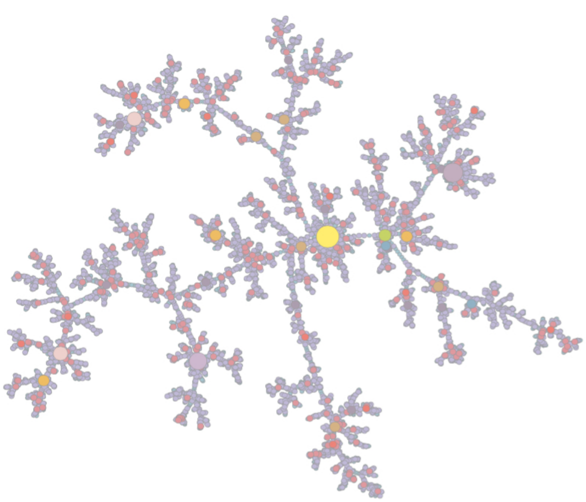

FIGURE 1.1. Inhomogeneous continuum random tree.

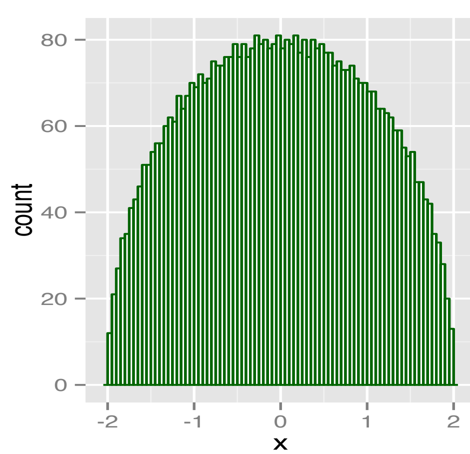

FIGURE 1.2. Empirical spectral distribution of a symmetric Gaussian random matrix. Enter ze semi-circular law!

this converges to a local neighborhood of an infinite (model dependent) random structure. This "local convergence" is amazingly enough to understand asymptotics for global functionals such as the partition function of graphical models or the spectral distribution of the adjacency matrix of a random graph. This technique has been used to great effect both in compressed sensing [11, 15], high-dimensional statistics [6], combinatorial optimization [3, 4] and statistical physics [10].

2. Topics covered

2.1. **General theory.** The first aim of this course is to understand general probability theory for convergence of sequences of random objects in nice enough spaces. To start with we will largely follow [8]. Topics that will be covered include:

(i) General theory of weak convergence.
(ii) Special cases: $\mathbb{R}$, $\mathbb{R}^\infty$ and $C[0,1]$. Specialized techniques in $\mathbb{R}$ including the method of moments and Stein's method.
(iii) Continuous mapping theorem.
(iv) Prokhorov's theorem and relative compactness. Skorohod representation theorem.
(v) Analysis of $C[0,1]$. Introduction to Brownian motion and related processes (including the Brownian bridge).
(vi) Martingale functional central limit theorem [12]. Applications in probabilistic combinatorics and applied probability [17].
(vii) A brief introduction to $D[0,1]$.
(viii) *If time permits:* A quick introduction to empirical processes theory.
(ix) *If time permits:* Basic convergence results to other processes including Levy type processes.
(x) *If time permits:* Wigner's semi-circular law and random matrices.

2.2. **Local weak convergence.** The second aim of the course is to develop the basics of local weak convergence. The eventual goal is to understand how this is used at the research level. Topics that will be **hopefully** be covered include:

(i) General foundational theory [3, 7].
(ii) Applications in probabilistic combinatorics and optimization [3].
(iii) Belief propagation: intuition and motivation [14].
(iv) Rigorous theory with applications in compressed sensing [11, 15] and statistical physics [10].
(v) Spectral distribution of random adjacency matrices [9].

3. Pre-requisites

A graduate level course in probability or permission from the instructor.

4. References

No book will be required for the course. For the first "half" of the course we will follow large parts of [8] however I will be using a number of other books including [12, Chapter 8]. For the second part of the course, see the references cited in Section 2.2 for a starting point.

5. Grading scheme

I will assign homework problems on Sakai. These will not be collected, however important problems will be discussed in class after you have had a chance to attempt the problems. In terms of the grade break down:

(i) 40%: **Latexing class material:** I will upload a "Lecture template". At the end of every class, the material covered on that day will be assigned to a student to Latex using the template on Sakai.

(ii) 40%: **Final report:** As the semester progresses chat with me and decide on a paper that you want to read that is related to the course. I am happy to suggest papers as well and will upload a number of papers onto Sakai. At the end of the semester, write a report (at least 5 pages, doublespaced) on the paper, the motivations of the authors **and** the kind of research questions the paper suggests for you. The aim of this report is to help you develop one of the most important tools in life: generating questions that interest your curiosity! **You are allowed to work in teams of at most two.**

(iii) 20%: **Class presentation:** On the last 2 class days of the semester, present a 20 minute talk on what you learnt whilst reading the paper you chose. Again if you are working in a team then the entire team does **one** presentation.

6. Administration details

- **Instructor email:** bhamidi@email
- **Office:** 304 Hanes Hall
- **Office Hours:** Tentatively 11-12 on Wednesday and 11-12:20 on Thursday.

There is no final for this class.

References

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[4] D. J Aldous, *The ζ (2) limit in the random assignment problem*, Random Structures & Algorithms **18** (2001), no. 4, 381–418.

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